# MARKOV RANDOM FIELDS AND GIBBS RANDOM FIELDS†

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ABSTRACT

Spitzer has shown that every Markov random field (MRF) is a Gibbs random field (GRF) and vice versa when (i) both are translation invariant, (ii) the MRF is of first order, and (iii) the GRF is defined by a binary, nearest neighbor potential. In both cases, the field (iv) is defined on $Z^v$, and (v) at any $x \in Z^v$, takes on one of two states. The current paper shows that a MRF is a GRF and vice versa even when (i)–(v) are relaxed, i.e., even if one relaxes translation invariance, replaces first order by $k$th order, allows for many states and replaces finite domains of $Z^v$ by arbitrary finite sets. This is achieved at the expense of using a many body rather than a pair potential, which turns out to be natural even in the classical (nearest neighbor) case when $Z^v$ is replaced by a triangular lattice.

## 1. Introduction

In a recent paper [6], Spitzer pursued certain notions of Dobrusin [3] where the one-dimensional discrete time of a Markov chain is replaced by the $v$-dimensional discrete time of a Markov random field (MRF). Instead of the past of a point being one-sided, it is now the complementary set to the point. Thus even when $v = 1$, a question arises as to the identity of the two concepts. Nevertheless, for the case of a MRF subject to the requirements:

(0) there are only two states (occupied and unoccupied),

(1) for every configuration $\omega$, $P(\omega) > 0$,

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2) there is translation invariance for the conditional probability,
3) a first order one Markov condition holds,

Spitzer showed that the conditional probabilities that arise are given by a binary, translation invariant, nearest neighbor Gibbs potential. (Details on these notions as well as conditions 0)–3) will be given in the body of the paper.) While it is trivial that such a Gibbs potential yields the conditional probabilities of a MRF satisfying 0) – 3) (and as a matter of fact it is this that suggested to Dobrusin the notion of a MRF), it is very satisfying to know that, conversely, every MRF comes from a Gibbs potential since no such physical notion is mentioned in the purely probabilistic notion of a MRF. Here one has a particularly happy marriage of ideas from probability and physics.

Spitzer had been trying to get an analogous theorem for the $k$th order MRF (definitions later) but his arguments for the first order case had an intricate geometric character which, although elegant in the latter case, ran aground in the $k$th order case.

In this paper we consider a MRF where there are a finite number of states not just two, where translation invariance is not assumed, and the first order Markov condition is replaced by a $k$th order Markov condition (in other words, we relax conditions 0), 2), 3)); we find that here also the conditional probabilities are given by a Gibbs potential but this time the interaction need not be translation invariant and we must allow *many-body* interaction. Thus the happy marriage of ideas from probability and physics continues. A simplification of the proof even for Spitzer's original case results. The burden is placed on repeated use of the principle of inclusion and exclusion rather than geometrical argument.

That the many body interaction is "right" can be seen even in Spitzer's original case for $\nu = 2$. If we retain the first order Markov condition but replace the square lattice for discrete time by a triangular lattice for discrete time, then pair interactions no longer suffice to yield the conditional probabilities. At least 3-body interactions must be permitted.

Ideas used in the proof yield a simple proof of the known result [6] that, for $\nu = 1$, the notions of MRF and Markov chain coincide if 1) is satisfied.

After some definitions from Spitzer's paper and routine terminology on many-body interactions, the exposition will proceed from simple concrete cases to more general cases. For convenience the order of exposition in Spitzer will be followed.

The proof of his main theorem uses three steps of which the first is trivial and the second involves the aforementioned elegant argument. In this paper the first trivial step is repeated mainly to establish notation. The novelty of this paper comes in the second step. Spitzer's third step is used here without elaboration.

## 2. Definitions

The definition of a random field (RF) will agree with that of Spitzer with the trivial exception that the connectedness of the domain is not required since it is not used in the proof. For concreteness, the relevant definitions are repeated. Let $Z^\nu$ be the lattice points in $\nu$-dimensional space. For $x, y$ in $Z^\nu$, $|x - y|$ is the Euclidean distance between $x$ and $y$. By a domain $D$ is meant a finite subset of $Z^\nu$. The boundary $\partial D$ of domain $D$ is $\{y \in Z^\nu \setminus D : (\exists x \in D) |x - y| = 1\}$. Use $\partial x$ as an abbreviation for $\partial\{x\}$. Denote $D \cup \partial D$ by $\bar{D}$. Use $\bar{x}$ as an abbreviation for $\overline{\{x\}}$. If $\Omega = \{0,1\}^D$, $\mathscr{F}$ is the collection of all subsets of $\Omega$, and $P$ is a probability measure on $\mathscr{F}$, then $(\Omega, \mathscr{F}, P)$ is called a random field (RF) on domain $D$. Thus we may think of a RF on $D$ as a probability measure on the set of all configurations of particles on $D$ where the configuration of particles described by $\omega \in \Omega$ is

$$A = \{x \in D : \omega(x) = 1\}. \tag{1}$$

The definition used here for a Gibbs random field (GRF) differs from that of Spitzer insofar as the interaction is neither required to be binary nor homogeneous. The notation for potential and energy will be that of [5, p. 21]. A function $\Phi$ from the finite subsets of $Z^\nu$ to $R$ (the reals) is called a potential. For each finite $X \subset Z^\nu$, let the energy

$$U(X) = \sum_{A \subset X} \Phi(A). \tag{2}$$

Suppose we assume in addition

$$(\forall x \in Z^\nu)(\forall E \subset \partial x)(\forall H \subset \bar{D} \setminus \bar{x})\ H \neq \varnothing \Rightarrow \Phi(\{x\} \cup E \cup H) = 0. \tag{*}$$

It should be remarked that when (*) is satisfied $\Phi(B) \neq 0 \Rightarrow \# B \leqq 2$. This remark fails when the lattice $Z^\nu$ is replaced by a plane triangular lattice with nearest neighbor distance equal to 1.

By a boundary value (BV) function is meant a map $\varphi : \partial D \to \{0,1\}$. It is convenient to extend $\omega \in \Omega$ to a map $\bar{\omega} : \bar{D} \to \{0,1\}$ so that

$$\bar{\omega}(x) = \begin{cases} \omega(x), & x \in D, \\ \varphi(x), & x \in \partial D. \end{cases}$$

Let $A$ be given by (1) and

(3) $$F = \{x \in \partial D : \varphi(x) = 1\}.$$

Then $\{x : \bar{\omega}(x) = 1\} = A \cup F$.

Suppose we are given a domain $D \subset Z^{\nu}$ and potential $\Phi$ satisfying (*). Then RF $(\Omega, \mathscr{F}, P)$ on $D$ is a *GRF with potential $\Phi$ and BV function $\varphi$ if $P$ is defined by the formula

(4) $$P(\omega) = Z^{-1}\exp[[-U(A \cup F)], \qquad \omega \in \Omega$$

$A$ and $F$ are given by (1) and (3) with $Z$ being the normalizing constant so that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

With the above $A$, we also write $P(A)$ instead of $P(\omega)$. In particular if $\varphi \equiv 0$ on $\partial D$, we get the *GRF with BV zero given by

(5) $$P(A) = Z^{-1}\exp[-U(A)], \qquad A \subset D.$$

If $D$ is a rectangle with opposite points identified, it produces a lattice torus $T$ without boundary. Here the *GRF on $T$ is called a periodic *GRF and its probability measure $P$ is defined by (5) with $D$ replaced by $T$.

Our definition of a Markov random field (MRF) agrees with that of Spitzer except that we do not require translation invariance and we use a slight variation in notation. As in the case of *GRF, we assume a given BV function $\varphi$: $\partial D \to \{0,1\}$ and we get different MRF's for different $\varphi$'s. When $D$ is replaced by $T$, we get a periodic MRF. Now a RF $(\Omega, \mathscr{F}, P)$ on $D$ will be called a MRF if it satisfies the two conditions (a), (b) below. First, we have

(a) $$(\forall \omega \in \Omega) \quad P(\omega) > 0$$

which we can write

$$(\forall A \subset D) \quad P(A) > 0.$$

From (a) we can define one-point conditional probabilities

(6) $$P[\omega(x) = 1 \mid \bar{\omega}(\cdot) = e(\cdot) \text{ on } \bar{D}\backslash\{x\}](\forall x \in D),$$

by the usual quotient. Note that $e: \bar{D}\setminus\{x\} \to \{0,1\}$ is arbitrary except that $e = \varphi$ on $\partial D$. If

$$E = \{x \in \partial x : e(x) = 1\} \subset \partial x \tag{7}$$

$$H = \{x \in \bar{D}\setminus\bar{x} : e(x) = 1\} \subset \bar{D}\setminus\bar{x}$$

and $A$ is as in (1), then we write (6) as

$$P[\{x\} \mid E \cup H]. \tag{8}$$

The second condition defining a MRF is

(b) $P[\{x\} \mid E \cup H]$ is independent of $H \subset \bar{D}\setminus\bar{x}$.

If $(\Omega, \mathscr{F}, P)$ is a RF on domain $D$ which satisfies (a), (b), then we say $(\Omega, \mathscr{F}, P)$ is a MRF on $D$ with BV function $\varphi$ (or a periodic MRF on $D = T$ when $D$ is made into a torus $T$ without boundary).

## 3. The main theorem

*Main Theorem. Every MRF on a domain D with BV function $\varphi$ is a *GRF with BV function $\varphi$ and vice versa.The same statement holds for periodic random fields.*

We fix domain $D \subset Z^{\nu}$ and BV function $\varphi$ and hence $F$ by (3). The periodic case can be handled by the same method. Step 1 of the proof will show that every *GRF is a MRF. This is trivial and will be done as in Spitzer but using the notation developed. Step 2 of the proof will show that for every MRF there exists a *GRF with the same conditional probabilities as the MRF. The main novelty of this paper lies in its treatment of Step 2.

*Step* 1. (Every *GRF is a MRF). Start with a *GRF whose probability measure $P$ is given by (5) and go on to verify (a) and (b). Condition (a) is obvious. To check (b) use (8) subject to (7). Suppose that $(E \cup H) \cap \partial D = F$ of (3). Then $(\forall x \in D)$

$$P[\{x\} \mid E \cup H] = \frac{P[\{x\} \cup E \cup H]}{P[\{x\} \cup E \cup H] + P[E \cup H]}$$

$$\frac{\exp(-\sum_{R \subset E, S \subset H} \Phi(\{x\} \cup R \cup S) - \sum_{R \subset E, S \subset H} \Phi(R \cup S))}{\exp(-\sum_{R \subset E, S \subset H} \Phi(\{x\} \cup R \cup S) - \sum_{R \subset E, S \subset H} \Phi(R \cup S)) + \exp - \sum_{R \subset E, S \subset H} \Phi(R \cup S)}$$

$$= \frac{1}{1 + \exp \sum_{R \subset E, S \subset H} \Phi(\{x\} \cup R \cup S)}.$$

Since $\Phi$ is the potential for a *GRF

$$\emptyset \neq S \subset \bar{D}\backslash\bar{x} \Rightarrow \Phi(\{x\} \cup R \cup S) = 0.$$

Thus

$$P[\{x\} \,|\, E \cup H] = P[\{x\} \,|\, E]$$

and we have MRF which completes Step 1.

*Step* 2. (Existence of a *GRF with the same conditional probabilities as given MRF). Note that by suitable choice of $\Phi(\emptyset)$ we can arrange that the $Z$ of equation (5) is one. If that is done and we use (1) then

$$-\log P(\omega) = U(A \cup F) = U^F(A) = \sum_{B \subset A} \Phi^F(B).$$

The potential $\Phi^f$ is defined on subsets of $D$ but it can be trivially extended to a potential defined on finite subsets of $Z^v$ and yielding the same $P$. Now suppress the superscript F. From the last equation and by the principle of inclusion and exclusion

$$(\forall A \subset D)\Phi(A) = \sum_{B \subset A} (-1)^{\# A\backslash B} U(B).$$

We can extend $\Phi$ to all finite subsets of $Z^v$ by requiring

$$(\forall R \neq R \cap D)\Phi(R) = 0\,.$$

Define

$$\Phi_{x;E}(H) = \sum_{R \subset E} \Phi(\{x\} \cup R \cup H) \tag{9}$$

$$U_{x;E}(I) = \sum_{H \subset I} \Phi_{x;E}(H). \tag{10}$$

If we use the MRF requirement and reverse the argument in Step 1, we see that if $E \subset \partial x$, then $U_{x;E}(I)$ is independent of $I$ for all $I \subset \bar{D}\backslash\bar{x}$. Note that

$$U_{x;E}(\emptyset) = \Phi_{x;E}(\emptyset) = \sum_{R \subset E} \Phi(\{x\} \cup R).$$

By applying inclusion-exclusion to (10) we see that if

$$\emptyset \neq I \subset \bar{D}\backslash\bar{x}, \tag{11}$$

then

$$\Phi_{x;E}(I) = \sum_{S \subset I} (-1)^{\# I\backslash S} U_{x;E}(S) = 0 \tag{12}$$

or equivalently if (11), then

(13) $$\sum_{R \subset E} \Phi(\{x\} \cup R \cup I) = 0.$$

By applying inclusion-exclusion to (13) we see that if (11) and $E \subset \partial x$ then

(14) $$\Phi(\{x\} \cup E \cup I) = 0$$

which is the *GRF condition so that the conditional probabilities are those of a *GRF and Step 2 is completed.

Now apply the argument of Spitzer's step 3 to see that conditional probabilities of the form (8) for all $x \in D$, $E \subset \partial x$, and $H \subset \bar{D} \backslash \bar{x}$ completely determines the RF which is in this case a MRF. His argument in his step 3 does not use translation invariance. This completes the proof of our main theorem. Note that no translation invariance was assumed and no uniqueness for the potential results.

## 4. Translation invariance

We can now consider what happens when we introduce the condition of translation invariance. We say that RF $(\Omega, \mathscr{F}, P)$ is translation invariant (t.i.) if

$$(\forall x \in D)(\forall E \subset \partial x)(\forall z \in Z^{\nu}) x + z \in D \Rightarrow P(\{x + z\} \mid E + z) = P(\{x\} \mid E).$$

We say that a potential is translation invariant if for all finite $A \subset Z^{\nu}$ and for all $z \in Z^{\nu}$

$$\Phi(A) = \Phi(A + z).$$

In an obvious way we can talk of a t.i. MRF and *GRF given by a t.i. potential. It is trivial that a *GRF given by a t.i. potential is a t.i. MRF so that the analogue to step 1 of Spitzer follows. The analogue of step 2 of Spitzer goes through if we assume

(15) $$(\exists x \in D)\, \bar{x} \subset D.$$

If this is done, the argument defines a $\Phi$ such that

$$(\forall E \subset \partial x)(\forall H \subset \bar{D} \backslash \bar{x})\ \Phi(\{x\} \cup E \cup H) = 0$$

and since we started with a t.i. MRF

$$A, B \subset \bar{x}\ \ (\forall y \in Z^{\nu})\ A = B + y : \Rightarrow : \Phi(A) = \Phi(B).$$

Thus $\Phi$ is defined for $\{\{x\} \cup E : E \subset \partial x\}$ and $\Phi$ is "locally t.i.". We extend $\Phi$ to $\{\{x + y\} \cup (E + y) : E \subset \partial x,\ y \in Z^{\nu}\}$ by $\Phi(B + y) = \Phi(B)$. This yields a t.i. *GRF which completes the analogue of Spitzer's step 2. There is no difficulty with step 3.

COROLLARY 1. *If* (15) *is satisfied for domain D every t.i. MRF on domain D with BV function* $\varphi$ *is a t.i. *GRF with BV function* $\varphi$ *and vice versa. The potential is defined uniquely up to an additive constant. The last two statements hold also for periodic random fields.*

When condition (15) is violated, the $D$ is too small for t.i. to play a really satisfactory role and the uniqueness no longer holds. Consider the following example: $\nu = 2$, $D = \{x\}$ and $BV$ function such that $\varphi(x+(0,1))=\varphi(x+(0,-1)) =1$ while $\varphi(x+(1,0))=\varphi(x+(-1,0))=0$. In this case there are many t.i. $\Phi$ 's that yield the MRF. In particular $\Phi$ for a horizontal nearest neighbour pair is arbitrary. This example shows that Spitzer's main theorem requires some condition like (15) in order to be correct.

## 5. Generalizations (including $k$-th order Markov)

Let us now go back to the ideas of Section 3. In that section

$$\partial x = \{y \in Z^\nu \backslash \{x\} : |x-y| = 1\}.$$

This may be replaced by allowing $\partial$ to be a mapping from elements of $L$ to subsets of $L$ not ontaining $x$ and let $\bar{x} = \{x\} \cup \partial x$ and $\bar{A} = \cup_{x \in A} \bar{x}$. $L$ need not be a lattice although in the previous case $L$ was $Z^\nu$. Let $D$ be a finite subset of $L$. We define a GRF with BV function $\varphi$ exactly as before. Now a *GRF is defined as before with new $\partial$ and bar. In the definition of a $\partial$ MRF we again replace in the definition of a MRF the old $\partial$ and bar by the new $\partial$ and bar. The argument presented in Section 3 goes through without change and we get

COROLLARY 2. *Every* $\partial$ *MRF on a domain D with BV function* $\varphi$ *is a *$\partial$ GRF with BV function* $\varphi$ *and vice versa. The same statement holds for periodic random fields.*

The condition on the $\Phi$ of our *$\partial$ GRF may be symmetrically formulated as for all finite $A \subset L$

$$x, y \in A \ \& \ y \in L \backslash \bar{x} \Rightarrow \Phi(A) = 0 \tag{16}$$

and Corollary 2 may be formulated as (16)$\Leftrightarrow\partial$ MRF.

One way of describing what we have done is to note that the operation $\partial$ defines a directed graph so that the edge $(x, y)$ is in the graph if and only if $y \in \partial x$. Corollary 2 is the analogue of our main theorem when the directed graph replaces the graph of the lattice $Z^\nu$.

Suppose we go back to $Z^\nu$ and let $y \in \partial x$ if $y \neq x$ and $y$ can be reached from $x$

in less than or equal to $k$ unit steps. Then the $\partial$ MRF condition is a $k$th order Markov condition and the specialization of (16) becomes the $k$th order Markov condition which Spitzer was seeking.

Another example would be the following. Let $Z^\nu = L$ and let $y \in \partial x$ mean that $y \neq x$ and $|x - y| \leqq k$. Other metrics than the Euclidean metric can be used.

Let us consider $\nu = 2$ and $L$ a triangular lattice with unit edge length in $\mathbb{R}^2$. For each $x \in L$, let $\partial x$ be the set of nearest neighbors. Thus for each $x \in L, \# \partial x = 6$. For $x, y \in L$, let $d(x, y)$ be the minimum of the number of nearest neighbor steps from $x$ to $y$. Note that the intermediate steps must be in $L$ but need not be in $A$. For finite subset $A$ of $L$, let $\delta(A)$, the diameter of $A$ be the $\max_{x,y \in A} d(x, y)$. Thus $\delta(\partial x) = 2$. The vertices of a triangle of the triangular lattice constitute a set $A$ such that $\delta(A) = 1$. For the $\partial$ we are now considering, condition (16) amounts to requiring

$$(\forall B)\delta(B) > 1 \Rightarrow \Phi(B) = 0. \tag{17}$$

Since the set $A$ of the next to the last sentence has three elements and diameter equal to one, we may have a *GRF with $\Phi(A) \neq 0$. That is, for a triangular lattice, the MRF is equivalent to *GRF which allows for 3-body interactions not merely 2-body interactions. It is this example to which reference was made in Section 1.

## 6. Markov chains

Let us consider $\nu = 1$, i.e., $Z = L$, and two $\partial$ functions:

$$(\forall x \in Z) \quad \partial_1 x = \{x - 1\}$$

and

$$(\forall x \in Z) \quad \partial_2 x = \{x - 1, x + 1\}.$$

Let $Rx$, the remote past of $x$ be $\partial\partial x \cup \partial\partial\partial x \cup \cdots \backslash \bar{x}$. For $\partial_1$, the $R_1 x$ is $\{z : z < x - 1\}$. For $\partial_2$ the $R_2 x$ is $Z \backslash \bar{x}$. Then the Markov chain condition on RF satisfying (a) of Section 2 can be formulated as

$$(\forall x \in D)(\forall E \subset \partial_1 x)(\forall H \subset R_1 x) P(x \mid E \cup H) = P(x \mid E). \tag{18}$$

The argument of Section 3 shows that (18) is satisfied if and only if the RF is a $*\partial_1$GRF. This in turn is true, if and only if $\Phi(A)$ is zero if $\# A > 2$ or if $\# A = 2$ (so $A = \{y_1, y_2\}$) and $|y_1 - y_2| > 1$). From the Main Theorem, this is true if and only if the RF is a $*\partial_2$GRF. We thus see that the $\partial_2$ MRF condition and the

Markov chain condition are equivalent. We could have replaced (18) by

$$(19)\qquad (\forall x \in D)(\forall E \subset \partial_1 x)(\forall H \subset Z \setminus (x \cup \partial_1 x) P(x \mid E \cup H) = P(x \mid E)$$

the resulting RF's would be the class $\partial_2$MRF's. This shows the equivalence [6] of the Markov chain condition and MRF condition when $\nu = 1$.

## 7. Many states

Hitherto we have considered the case where there are only two states, unoccupied or occupied, i.e., $(\forall x \in D)\ \omega(x) \in \{0,1\}$. Now suppose we return to the context of Sections 2 and 3. We generalize this to allow for more states. We can handle the case where the number of states varies with the $x \in D$ but the idea is clear if we analyze the case where $(\forall x \in D)\omega(x) \in \{0,1,2,\cdots,s\} = S$ and $s$ does not depend on $x$. In Sections 3 and 4, we have considered the case $s = 1$. In order to save notation we will only analyze the periodic case.

Now $\Omega = S^D$. $\mathscr{F}$ and $P$ are as before. Assume condition a) is satisfied. Consider $\alpha, \beta \in \Omega$. We define $\beta \leqq \alpha$ to mean

$$(\forall x)\beta(x) \in \{0, \alpha(x)\}.$$

To get a GRF we need the energy. If $\alpha \in \Omega$ then $U(\alpha)$ is real valued. Before, when $s = 1$ and $X = \{x : \omega(x) = 1\}$, we talked of $U(X)$ and the analogue of what we do now would have been to talk of $U(\omega)$. Now we let

$$U(\alpha) = \sum_{\beta \leqq \alpha} \Phi(\beta)$$

and we can define $\Phi$ by inclusion-exclusion. The analogue for many states to (*) is

$$(**)\qquad (\forall x \in Z^\nu)(\forall a \in S) a \neq 0 \ \ \& \alpha(x) = a \ \& \ \ \alpha \mid T \setminus \bar{x} \not\equiv 0 \Rightarrow \Phi(\alpha) = 0.$$

Since we are in the periodic case, we need not consider boundary values and a ** GRF with potential $\Phi$ (satisfying (**)) if $P$ is defined by the analogue to (5).

In the many state case of MRF, condition (a) of Section 2 is required. The analogue to condition (b) is for all nonzero $a \in S$

$$(b')\qquad P(\omega(x) = a \mid \omega(\cdot) = \alpha(\cdot) \text{ on } D \setminus \{x\})$$

is independent of $\alpha$ on $T \setminus \bar{x}$. If $(\Omega, \mathscr{F}, P)$ is periodic RF (with many states) on $T$ satisfying (a) and (b′), then we say that $(\Omega, \mathscr{F}, P)$ is a periodic MRF on $T$.

COROLLARY 3. *Every periodic MRF with set of states S on T is **GRF and vice versa.*

The analogue of Step 1 of the Main Theorem goes through without complications. Condition (**) $\Rightarrow$ (b′).

For the analogue to Step 2, we start with a many state MRF, i.e., a RF satisfying (a) and (b′). On account of (b′) it is easy to show that for all nonzero $a \in S$

$$\text{(b}''\text{)} \qquad P(\omega(x) = a \,|\, \omega(\cdot) = \alpha(\cdot) \text{ on } D\setminus\{x\} \quad \omega(x) \in \{0, a\})$$

is independent of $\alpha$ on $T\setminus\bar{x}$. This in effect reduces the consideration to at most two states at each location with one of the states being 0 and the other being the value of $\alpha$ at the location. Now the argument of Step 2 can be applied to yield (**) so that a many state MRF has conditional probabilities given by a **GRF. Step 3 as before shows that the conditional probabilities of the form (b′) determine the RF.

Analogoues to Corollary 3 may be established for the general $\partial$. For the t.i. case where uniqueness is again attained if (15) is staisfied, one gets uniqueness of the potential $\Phi$ in the **GRF.

## 8. Discussion

The contents of the current paper, already compared (with that of Spitzer), should be compared with that of Averintsev [1] and that of [Hammersely and Clifford [4]. Averintsev allows for many states but limits himself to a binary potential and to $Z^\nu$ as the multidimensional time. He also does not deal with the $k$th order MRF. Hammersely and Clifford deal with the case where $Z^\nu$ is replaced by a general graph and so can be adapted to a $k$th order MRF although they do not do this. The repeated use of inclusion-exclusion in the current paper is represented by a somewhat more complicated "blackening algebra" in [4]. In [4] many states are considered as well as many body potentials.

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NOTE. After this paper was sent to the typist [2] appeared. It continues the original Averintsev paper [1] insofar as what is called $\partial$ MRF in this paper in a translation invariant setting with $L = Z^\nu$ and the details developed are very interesting. In particular, the $k$th order MRF is also considered. The methods used are different from those of this paper.

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